Calc 12 – Series. (from Stewart)

Note Title 2015-05-11

Chp 11.1 – lots of definitions and theorems, covering minimally because this is not the hard part!

Sequence : $\{a_1, a_2, \dots\}$ is also $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

Examples: $a_n = \dfrac{n}{n+1}$,

$a_n = \dfrac{(-1)^n(n+1)}{3^n}$,

Recursive Sequence: $f_n = f_{n-1} \dots$

1 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \to \infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n \to \infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

2 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n \to \infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$|a_n - L| < \varepsilon \qquad \text{whenever } n > N$$

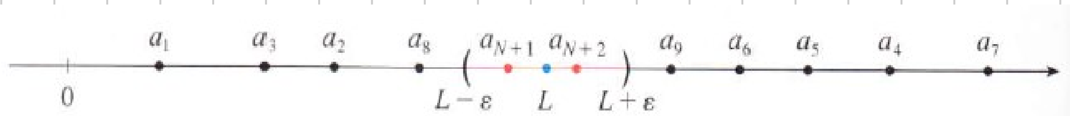

FIGURE 4

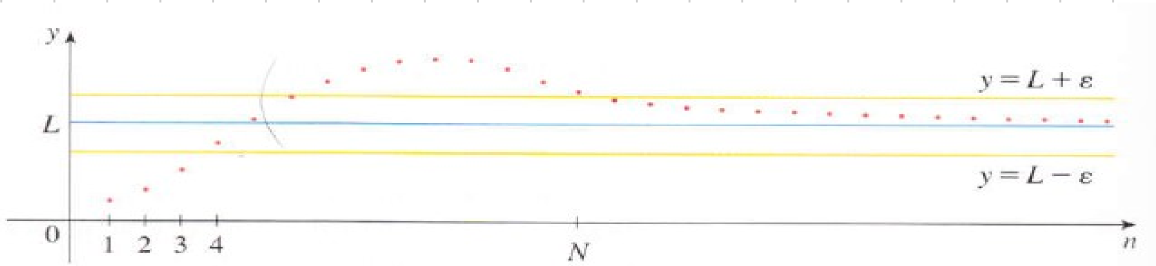

FIGURE 5

5 Definition $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$a_n > M \qquad \text{whenever } n > N$$

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

6 Theorem If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

If $-1 < r < 0$, then $0 < |r| < 1$, so

$$\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = 0$$

and therefore $\lim_{n\to\infty} r^n = 0$ by Theorem 6. If $r \leqslant -1$, then $\{r^n\}$ diverges as in Example 6. Figure 11 shows the graphs for various values of r. (The case $r = -1$ is shown in Figure 8.)

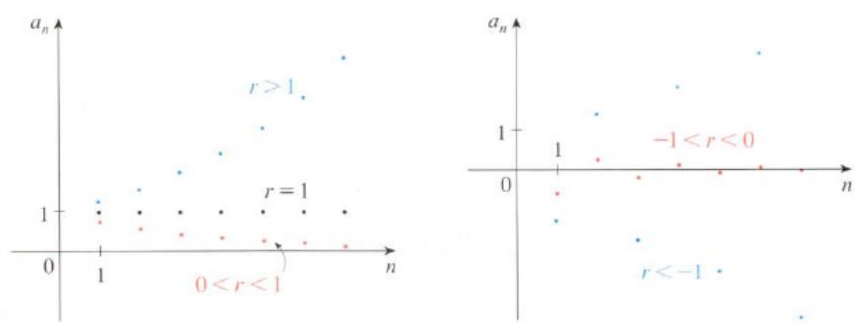

FIGURE 11
The sequence $a_n = r^n$

The results of Example 9 are summarized for future use as follows.

8 The sequence $\{r^n\}$ is convergent if $-1 < r \leqslant 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

9 Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geqslant 1$. It is called **monotonic** if it is either increasing or decreasing.

10 Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leqslant M \qquad \text{for all } n \geqslant 1$$

It is **bounded below** if there is a number m such that

$$m \leqslant a_n \qquad \text{for all } n \geqslant 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

11 Monotonic Sequence Theorem Every bounded, monotonic sequence is convergent.

Chp 11.2 - Series (Infinite Series)

$$\sum_{n=1}^{\infty} a_n \text{ or } \sum a_n = a_1 + a_2 + a_3 + \cdots$$

Most will be divergent, so not interesting - nothing to solve.

However: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots =$

2 Definition Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series. Otherwise, the series is called **divergent**.

4 The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

If $|r| \geqslant 1$, the geometric series is divergent.

Solving with different methods.
Telescoping series (terms cancel out):

$$S_n = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \cdots$$

Partial Fraction Decomp : $\frac{1}{a(a+1)} =$

Non-example : $\sum_{n=1}^{\infty} (-1)^n$

$S_n =$

Harmonic Series: $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

6 Theorem If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Converse is not true, eg Harmonic Series.

Converse: If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ is convergent

not true.

7 The Test for Divergence If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

seq diverges

seq converges not at 0.

contrapositive of theorem 6.

eg) Show $\sum_{n=1}^{\infty} \frac{n^3}{2n^3+6}$ is divergent.

8 Theorem If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum (a_n + b_n)$, and $\sum (a_n - b_n)$, and

(i) $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$ (ii) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Chp 11.3 - 11.7: Integral Test, p-Series, Comparison Test, Limit Comparison Test, Alternating Series Test, Absolute Convergence, Ratio Test, Root Test.

Chp 11.8: Power Series. Form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \quad \boxed{1}$$

If $c_n = 1$, $\forall n$ then:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

We can center then Power Series about $x=a$.

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots \quad \boxed{2}$$

eg) For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent? Use Ratio Test eg) For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n}$ converge? Use Ratio Test.

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3 Theorem For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

radius of convergence.

4 Possible solutions for convergence: (interval of convergence)

$[a-R, a+R]$ $(a-R, a+R)$

$[a-R, a+R)$ $(a-R, a+R]$ See above eg.

eg) Find the radius of convergence for $\sum_{n=0}^{\infty} (-2)^n x^n$.

11.9 Representation of Functions as Power Series.

Converting other functions to a Power Series:

Recall: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$, $|x| < 1$

eg) Find power series for $\frac{1}{1+x^2}$

eg) Find power series for $\frac{1}{x-3}$

eg) Find power series for $\frac{x^4}{x+4}$

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2 Theorem If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

NOTE 1 Equations (i) and (ii) in Theorem 2 can be rewritten in the form

(iii) $\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}[c_n(x-a)^n]$

(iv) $\int\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right]dx = \sum_{n=0}^{\infty}\int c_n(x-a)^n\,dx$

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eg) Find power series for $\frac{1}{(1-x)^2}$

Note: not the same as $\frac{1}{1-x}$ or $\frac{1}{1+x^2}$!

eg) Find a power series for $\ln(1-x)$

How do we estimate $\ln 2$?

Try to differentiate the function, then see if ~~you match~~ can match $\frac{1}{1-x}$, if so, then it is a power series.

Chp 11.10: Taylor & Maclaurin Series

Recall linear approx

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots \quad |x-a| < R \tag{1}$$

$$f(a) = c_0$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \quad |x-a| < R \tag{2}$$

$$f'(a) = c_1$$

$$f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + \cdots \quad |x-a| < R \tag{3}$$

$$f''(a) = 2c_2$$

$$f'''(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x-a) + 3 \cdot 4 \cdot 5c_5(x-a)^2 + \cdots \quad |x-a| < R \tag{4}$$

$$f'''(a) = 2 \cdot 3c_3 = 3!c_3$$

$$c_n = \frac{f^{(n)}(a)}{n!}$$

linear approx.

higher order approx.

$$\boxed{6} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

The series in Equation 6 is called the **Taylor series of the function *f* at *a*** (or **about *a*** or **centered at *a***). For the special case $a = 0$ the Taylor series becomes

simplifies.

$$\boxed{7} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

This case arises frequently enough that it is given the special name **Maclaurin series**.

eg) Find the Maclaurin Series for $f(x) = e^x$. Note this does not necessarily mean that it works. until we do a test.

Radius of Convergence:

$$a_n = \frac{x^n}{n!}$$

Ratio Test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{|x|}{n+1} < 1, \text{ so } R = \infty \ \forall x$$

Proof that this power series actually represents e^x is very long so we need to skip.

Let $x = 1$

$$e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots = 2.7182\ldots$$

eg) Find the Maclaurin Series for $f(x) = \sin x$

eg) Find the Maclaurin Series for $f(x) = \cos x$

That's it! Have a great summer!

$i = \sqrt{-1} = i$
$i^2 = \sqrt{-1}^2 = -1$
$i^3 = \sqrt{-1}^3 = -i$
$i^4 = \sqrt{-1}^4 = 1$
$i^5 = \sqrt{-1}^5 = i$

So what is e^{ix} as a power series?